SKEWED DISTRIBUTIONS

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The Student-t distribution

Definition

A random variable Y has a Student-t distribution with mean parameter μ, scale parameter σ^2 and degree of freedom ν, denoted by $\mathrm{t}(\mu, \sigma^2; \nu)$, if it has the following stochastic representation:

$$Y = \mu + U^{-1/2} Z, \;\; U \perp Z,$$

where $Z \sim N(0, \sigma^2)$ and $U \sim Gamma(\nu/2, \nu/2)$.

- The pdf of Y is given by

$$t(y|\mu, \sigma^2, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma} \left(1 + \frac{(y-\mu)^2}{\sigma^2 \nu}\right)^{-\frac{\nu+1}{2}}.$$

- The Student-t distribution has the normal distribution as limiting case as $\nu -> \infty$.
- $E[Y] = \mu$ and $Var(Y) = \sigma^2 \frac{\nu}{\nu-2}$
- Slash distribution if $U \sim Beta(\nu, 1)$. If U is a positive random variable with pdf (or pmf) $h(u|\boldsymbol{\nu})$, we get the scale mixtures of normal (SMN) distributions family.

The univariate skew-normal distribution

Definition

A random variable Z follows a skew-normal distribution with location parameter μ, scale parameter σ^2 and skewness parameter λ, denoted by $\mathrm{SN}(\mu, \sigma^2, \lambda)$, if its pdf is given by (Azzalini, 1985)

$$SN(y|\mu, \sigma^2, \lambda) = 2\phi(z|\mu, \sigma^2)\Phi(\frac{z-\mu}{\sigma}\lambda),$$

where $\phi(.|\mu, \sigma^2)$ is the pdf and $\Phi(.)$ is the cdf of the univariate normal distribution.

- Notation: $Z \sim SN(\mu, \sigma^2, \lambda)$
- Stochastic representation **(useful to generate random numbers)**:

$$Z = \mu + \Delta|T_0| + \Gamma^{1/2}T_1,$$

  where $\Delta = \sigma\delta$, $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$, $\Gamma = (1-\delta^2)\sigma^2$, T_0 and T_1 are independent standard normal random variables.
- when $\lambda = 0$, the SN distribution reduces to $N(\mu, \sigma^2)$.
- $E[Z] = \mu + \sqrt{2/\pi}\Delta$ and $var(Z) = \sigma^2(1 - 2/\pi\delta^2)$

The univariate SMSN class of distributions

Definition

A random variable Y has a SMSN distribution with location parameter μ, scale parameter σ^2 and skewness parameter λ, denoted by $\mathrm{SMSN}(\mu, \sigma^2, \lambda; H)$, if it has the following stochastic representation:

$$Y = \mu + \kappa^{1/2}(U)Z, \ \ U \perp Z,$$

where $Z \sim SN(0, \sigma^2, \lambda)$, U is a positive random variable with cdf $H(\,\cdot\,|\boldsymbol{\nu})$

- Mostly $\kappa(u) = 1/u$
- The pdf ofY is given by (letting $\frac{\lambda(y-\mu)}{\sigma}$)

$$\phi_{SMSN}(y|\mu, \sigma^2, \lambda, \boldsymbol{\nu}) = 2\int_0^\infty \phi(y|\mu, u^{-1}\sigma^2)\Phi\left(u^{1/2}A\right) dH(u|\boldsymbol{\nu}).$$

- When $\lambda = 0$, the SMSN family reduces to the symmetric class of scale mixtures of normal independent (SMN) distributions (Andrews and Mallows, 1974).
- $E[Y] = \mu + \sqrt{2/\pi}\Delta\kappa_1$ and $Var(Y) = \sigma^2(\kappa_2 - \frac{2}{\pi}\kappa_1\delta^2)$, with $\kappa_m = E[\kappa^{m/2}(U)]$

Particular cases

- **skew-normal distribution** when U is a degenerated r.v. at 1;
- **skew-t distribution** when $U \sim Gamma(\nu/2, \nu/2)$ ($Y \sim ST(\mu, \sigma^2, \lambda; \nu)$).

$$ST(y|\mu, \sigma^2, \lambda, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}\sigma}\left(1 + \frac{d}{\nu}\right)^{-\frac{\nu+1}{2}} T\left(\sqrt{\frac{\nu+1}{d+\nu}}A; \nu+1\right), \quad y \in \mathbb{R},$$

- **skew-slash distribution** when $U \sim Beta(\nu, 1)$ ($Y \sim SSL(\mu, \sigma^2, \lambda; \nu)$).

$$SSL(y|\mu, \sigma^2, \lambda, \nu) = 2\nu \int_0^1 u^{\nu-1}\phi(y; \mu, u^{-1}\sigma^2)\Phi(u^{1/2}A)du, \quad y \in \mathbb{R},$$

- **skew-contaminated normal distribution** when U is a discrete r.v. with pmf

$$h_{CN}(u|\nu, \gamma) \quad = \quad \nu I_{(u=\gamma)} + (1-\nu)I_{(u=1)}.$$

$$SCN(y|\mu, \sigma^2, \lambda, \nu, \gamma) = 2\{\nu\phi(y; \mu, \gamma^{-1}\sigma^2)\Phi(\gamma^{1/2}A) + (1-\nu)\phi(y; \mu, \sigma^2)\Phi(A)\}.$$

$$Y \sim SCN(\mu, \sigma^2, \lambda; \nu, \gamma)$$

The univariate SMSN class of distributions

Lemma

Let $Y \sim SMSN(\mu, \sigma^2, \lambda; H)$*. Then, the cdf of* Y *can be written in the following way:*

$$F_{SMSN}(y \mid \mu, \sigma^2, \lambda \; ; \; H) = \int_0^{\infty} 2\Phi_2\Big(\boldsymbol{y}(u)^* | \boldsymbol{\mu}^*, \boldsymbol{\Sigma}\Big) dH(u), \tag{1}$$

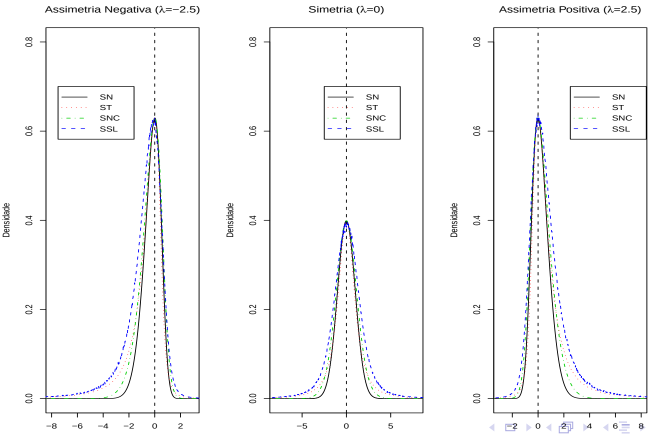
where

$$\boldsymbol{y}(u)^* = (\kappa(u)^{-1/2} y, 0)^\top, \quad \boldsymbol{\mu}^* = (\mu, 0)^\top, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma^2 & -\delta\sigma \\ -\delta\sigma & 1 \end{pmatrix} \tag{2}$$

and $\Phi_m(\cdot \mid \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ *denotes the cdf of the* $m-$*variate normal distribution with mean vector* $\boldsymbol{\mu}_0$ *and covariance matrix* $\boldsymbol{\Sigma}_0$*.*

- Mostly $\kappa(u) = 1/u$

Comparing SMSN densities



Application of the SMSN class in regression models

- The SMSN linear regression model. (Aldo Medina Garay, Master Dissertation, 2009)

$$Y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, , \varepsilon_i \sim SMSN(-\sqrt{\frac{2}{\pi}}k_1\Delta, \sigma^2, \lambda; H), i = 1, \ldots, n,$$

- The SMSN nonlinear regression model. (Aldo Medina Garay, Master Dissertation, 2009)

$$Y_i = \eta(\boldsymbol{\beta}, \mathbf{x}_i) + \varepsilon_i,\ \varepsilon_i \sim SMSN(-\sqrt{\frac{2}{\pi}}k_1\Delta, \sigma^2, \lambda; H),\ i = 1, \ldots, n,$$

- The SMSN censored linear regression model. (Thalita do Bem Mattos, Master Dissertation, 2016)

$$Y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \varepsilon_i, \quad i = 1, \ldots, n,$$

$$V_i = \begin{cases} c_i & \text{if} \quad \rho_i = 1 \text{ (i.e. } Y_i \le c_i); \\ Y_i & \text{if} \quad \rho_i = 0 \text{ (i.e. } Y_i > c_i), \end{cases}$$

R packages for fitting SMSN densities

- nlsmsn: Fitting univariate non-linear scale mixture of skew-normal regression models. (2012)
- FMsmsnReg: Regression Models with Finite Mixtures of Skew Heavy-Tailed Errors (2016)
- BayesCR: Bayesian Analysis of Censored Regression Models Under Scale Mixture of Skew Normal Distributions (2017)

The multivariate skew-normal distribution

Definition

we say that a p-dimensional random vector $\mathbf{Y}$ has a skew-normal distribution with location vector $\boldsymbol{\mu}$, scale matrix $\boldsymbol{\Sigma}$ and shape parameter vector $\boldsymbol{\lambda}$ if it has pdf given by (Azzalini and Dalla Valle, 1996)

$$f(\mathbf{y}) = 2\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi(\boldsymbol{\lambda}^\top \boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})), \quad \mathbf{y} \in \mathbb{R}^p.$$

We use the notation $\mathbf{Y} \sim \mathrm{SN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda})$.

- Stochastic representation:

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2}(\boldsymbol{\delta}|X_0| + (\mathbf{I}_p - \boldsymbol{\delta}\boldsymbol{\delta}^\top)^{1/2}\mathbf{X}_1), \quad \text{with} \quad \boldsymbol{\delta} = \frac{\boldsymbol{\lambda}}{\sqrt{1+\boldsymbol{\lambda}^\top\boldsymbol{\lambda}}}. \qquad (3)$$

  where $X_0 \sim N(0,1)$ and $T_1 \sim N_p(0, \mathbf{I}_p)$ are independent.
- when $\boldsymbol{\lambda} = 0$, the SN distribution reduces to $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
- $$\mathrm{E}[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta} \text{ and } \mathrm{Var}[\mathbf{Y}] = \boldsymbol{\Sigma} - \frac{2}{\pi}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^\top\boldsymbol{\Sigma}^{1/2}.$$

The multivariate SMSN class of distributions

Definition

A random vector Y has a SMSN distribution with location parameter $\boldsymbol{\mu}$, scale parameter $\boldsymbol{\Sigma}$ and skewness parameter $\boldsymbol{\lambda}$, denoted by $\mathrm{SMSN}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda; H)$, if it has the following stochastic representation:

$$Y = \boldsymbol{\mu} + \kappa^{1/2}(U)Z, \;\; U \perp Z,$$

where $Z \sim SN(0, \sigma^2, \lambda)$, U is a positive random variable with cdf $H(\cdot \,|\boldsymbol{\nu})$

- The pdf of Y is given by (letting $\frac{\lambda(y-\mu)}{\sigma}$)

$$f(\mathbf{y}) = 2\int_0^{\infty} \phi_p(\mathbf{y}|\boldsymbol{\mu}, u^{-1}\boldsymbol{\Sigma})\Phi(u^{1/2}\boldsymbol{\lambda}^{\top}\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu}))dH(u;\boldsymbol{\nu}), \qquad (4)$$

- When $\boldsymbol{\lambda} = 0$, the SMSN family reduces to the symmetric class of scale mixtures of normal (SMN) distributions (Andrews and Mallows, 1974).
- $E[U^{-1/2}] < \infty$, then $E[\mathbf{Y}] = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}E[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$ and $Var[\mathbf{Y}] = \boldsymbol{\Sigma}_y = E[U^{-1}]\boldsymbol{\Sigma} - \frac{2}{\pi}E^2[U^{-1/2}]\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}\boldsymbol{\delta}^{\top}\boldsymbol{\Sigma}^{1/2}$

Useful properties

Proposição

Let $\mathbf{Y} \sim SMSN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$. *Then for any fixed vector* $\mathbf{b} \in \mathbb{R}^m$ *and matrix* $\mathbf{A} : m \times p$ *of full row rank matrix,*

$$\mathbf{b} + \mathbf{A}\mathbf{Y} \sim SMSN_p(\mathbf{b} + \mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top, \boldsymbol{\lambda}^*; H),$$

where $\boldsymbol{\lambda}^* = \boldsymbol{\delta}^*/(1 - \boldsymbol{\delta}^{*\top}\boldsymbol{\delta}^*)^{1/2}$, *with* $\boldsymbol{\delta}^* = (\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)^{-1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\delta}$.

Corollary

Let $\mathbf{Y} \sim SMSN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ $\mathbf{Y}$ *be partitioned as* $\mathbf{Y}^\top = (\mathbf{Y}_1^\top, \mathbf{Y}_2^\top)^\top$ *of dimensions* p_1 *and* p_2 $(p_1 + p_2 = p)$, *respectively; let* $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1^\top, \boldsymbol{\mu}_2^\top)^\top$ *be the corresponding partitions of* $\boldsymbol{\Sigma}$ *and* $\boldsymbol{\mu}$. *Then*

$$\mathbf{Y}_1 \sim SMSN_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}, \boldsymbol{\Sigma}_{11}^{1/2}\widetilde{\boldsymbol{\upsilon}}; H),$$

where $\widetilde{\boldsymbol{\upsilon}} = \dfrac{\boldsymbol{\upsilon}_1 + \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\upsilon}_2}{\sqrt{1 + \boldsymbol{\upsilon}_2^\top \boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}}$. *with* $\boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$,

Useful properties

Proposição

Under the notation of Corollary 6, if $\mathbf{Y} \sim SMSN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$ *then the distribution of* $\mathbf{Y}_2$ *conditionally on* $\mathbf{Y}_1 = \mathbf{y}_1$ *and* $U = u$*, has density given by*

$$f(\mathbf{y}_2|\mathbf{y}_1, u) = \phi_{p_2}(\mathbf{y}_2|\boldsymbol{\mu}_{2.1}, u^{-1}\boldsymbol{\Sigma}_{22.1})\frac{\Phi(u^{1/2}\boldsymbol{\upsilon}^\top(\mathbf{y} - \boldsymbol{\mu}))}{\Phi(u^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))}, \qquad (5)$$

with $\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$.

Corollary

Consider the notation of Corollary 6. If $\mathbf{Y} \sim SMSN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}; H)$*, then the first moment of* $\mathbf{Y}_2$ *conditionally on* $\mathbf{Y}_1 = \mathbf{y}_1$ *is given by*

$$E[\mathbf{Y}_2|\mathbf{y}_1] = \boldsymbol{\mu}_{2.1} + \frac{\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}{\sqrt{1 + \boldsymbol{\upsilon}_2^\top\boldsymbol{\Sigma}_{22.1}\boldsymbol{\upsilon}_2}} E[U^{-1/2}\frac{\phi_1(U^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))}{\Phi_1(U^{1/2}\widetilde{\boldsymbol{\upsilon}}^\top(\mathbf{y}_1 - \boldsymbol{\mu}_1))}|\mathbf{y}_1],$$

with $\boldsymbol{\mu}_{2.1} = \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{y}_1 - \boldsymbol{\mu}_1)$.

Particular cases

- **Multivariate skew-t distribution** $\mathrm{ST}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ and $U \sim Gamma(\nu/2, \nu/2)$

$$f(\mathbf{y}) = 2\mathrm{t}_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu)\mathrm{T}_1\left(\frac{\sqrt{\nu+p}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})}{\sqrt{d+\nu}}|0, 1, \nu+p\right), \quad \mathbf{y} \in \mathbb{R}^p,$$

Proposição

If $\mathbf{Y} \sim \mathrm{ST}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$. *Then,*

$$E[U^r|\mathbf{y}] = \frac{2^{r+1}\nu^{\nu/2}\Gamma(\frac{p+\nu+2r}{2})(d+\nu)^{-\frac{p+\nu+2r}{2}}}{f(\mathbf{y})\Gamma(\nu/2)\sqrt{\pi^p}|\boldsymbol{\Sigma}|^{1/2}} T_1\left(\sqrt{\frac{p+\nu+2r}{d+\nu}}A|0, 1, p+\nu+2r\right),$$

and

$$E[U^r W_{\Phi_1}(U^{1/2}A)|\mathbf{y}] = \frac{2^{r+1/2}\nu^{\nu/2}\Gamma(\frac{p+\nu+2r}{2})(d+\nu+A^2)^{-\frac{p+\nu+2r}{2}}}{f(\mathbf{y})\Gamma(\nu/2)\sqrt{\pi}^{p+1}|\boldsymbol{\Sigma}|^{1/2}}.$$

where $A = \boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})$.

Particular cases

- **Multivariate skew-slash distribution** $SSL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$ and $U \sim beta(\nu, 1)$

$$f(\mathbf{y}) = 2\nu \int_0^1 u^{\nu-1}\phi_p(\mathbf{y}|\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{u})\Phi_1(u^{1/2}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})), \quad \mathbf{y} \in \mathbb{R}^p,$$

Proposição

If $\mathbf{Y} \sim SSL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu)$. *Then,*

$$E[U^r|\mathbf{y}] = \frac{2^{\nu+r+1}\nu\Gamma(\frac{p+2\nu+2r}{2})P_1(\frac{p+2\nu+2r}{2}, \frac{d}{2})d^{-\frac{p+2\nu+2r}{2}}}{f(\mathbf{y})\sqrt{\pi^p}|\boldsymbol{\Sigma}|^{1/2}}E[\Phi(S^{1/2}A)],$$

where $S_i \sim Gamma(\frac{p+2\nu+2r}{2}, \frac{d}{2})\mathbb{I}_{(0,1)}$, *and*

$$E[U^r W_{\Phi_1}(U^{1/2}A)|\mathbf{y}] = \frac{2^{\nu+r+1/2}\nu\Gamma(\frac{2\nu+p+2r}{2})}{f(\mathbf{y})\sqrt{\pi}^{p+1}|\boldsymbol{\Sigma}|^{1/2}}(d+A^2)^{-\frac{2\nu+p+2r}{2}}P_1(\frac{2\nu+p+2r}{2}, \frac{d+A^2}{2})$$

where $P_x(a, b)$ *denotes the cdf of the* $Gamma(a, b)$ *distribution evaluated at* x.

Particular cases

- **Multivariate skew-contaminated normal distribution** $SCN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$, $0 < \nu < 1$, $0 < \gamma < 1$ and U as in the univariate case

$$
\begin{aligned}
f(\mathbf{y}) &= 2\{\nu\phi_p(\mathbf{y}|\boldsymbol{\mu}, \frac{\boldsymbol{\Sigma}}{\gamma})\Phi_1(\gamma^{1/2}\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu})) \\
&\quad +(1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi_1(\boldsymbol{\lambda}^\top\boldsymbol{\Sigma}^{-1/2}(\mathbf{y}-\boldsymbol{\mu}))\},
\end{aligned}
$$

Proposição

If $\mathbf{Y} \sim SCN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\lambda}, \nu, \gamma)$. *Then,*

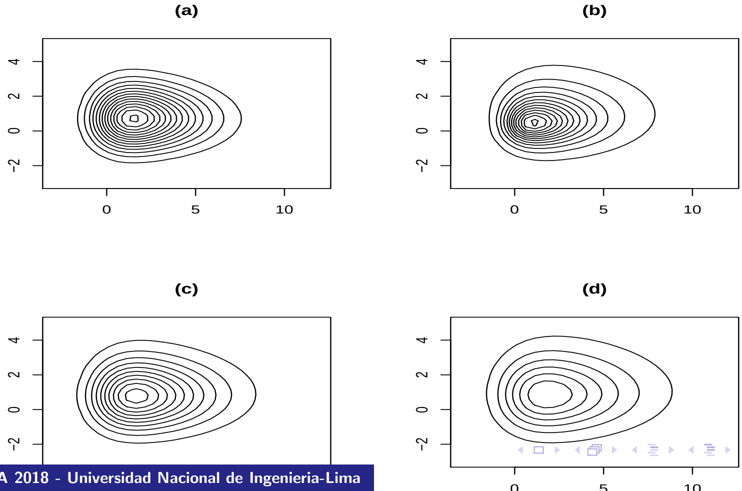
$$
E[U^r|\mathbf{y}] = \frac{2}{f(\mathbf{y})}[\nu\gamma^r\phi_p(\mathbf{y}|\boldsymbol{\mu}, \gamma^{-1}\boldsymbol{\Sigma})\Phi_1(\gamma^{1/2}A) + (1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\Phi_1(A)]
$$

and

$$
E[U^r W_{\Phi_1}(U^{1/2}A)|\mathbf{y}] = \frac{2}{f(\mathbf{y})}[\nu\gamma^r\phi_p(\mathbf{y}|\boldsymbol{\mu}, \gamma^{-1}\boldsymbol{\Sigma})\phi_1(\gamma^{1/2}A) + (1-\nu)\phi_p(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma})\phi_1(A)].
$$

Comparing SMSN contours

Figure: Contour plot of some elements of the standard bivariate SMSN family. (a) $\mathrm{SN}_2(\boldsymbol{\lambda})$ (b) $\mathrm{ST}_2(\boldsymbol{\lambda}, 2)$ (c) $\mathrm{SCN}_2(\boldsymbol{\lambda}, 0.5, 0.5)$ (d) $\mathrm{SSL}_2(\boldsymbol{\lambda}, 1)$, where $\boldsymbol{\lambda} = (2, 1)^\top$.



Some recent publications using SMSN distributions

- Lachos, Dey and Cancho (2009).[JSPI, Robust linear mixed models with skew-normal independent distributions from a Bayesian perspective.]
- Basso, Lachos, Cabral and Ghosh (2010).[CSDA, Robust mixture modeling based on scale mixtures of skew-normal distributions.]
- Lachos, Ghosh and Arellano-Valle (2010).[Sinica, Likelihood based inference for skew-normal/independent linear mixed model.]
- Garay, Lachos and Ortega and Vilca (2012).[JSPI, Estimation and diagnostics for heteroscedastic nonlinear regression models based on scale mixtures of skew-normal distributions.]
- Cabral, Lachos and Prates (2012).[CSDA, Robust multivariate mixture modelling using scale mixtures of skew-normal distributions.]
- Prates, Lachos and Cabral (2013).[Journal of Statistical Software, Fitting finite mixture of scale mixtures of skew-normal distributions.]
- Ferreira, Lachos and Bolfarine (2016).[Advances in Statistical Analysis, Multivariate skew scale mixtures of normal distributions.]
- Lachos, Cabral and Garay (2018). Moments of truncated scale mixtures of skew-normal distributions. [Brazilian Journal de Probability and Statistics]
- Padilla, Azevedo and Lachos (2018). Multidimensional multiple group IRT models with skew normal latent trait distributions[Journal of Multivariate Analysis]

Thank you!